The Parameter Estimation for the CIR model

Research conducted to facilitate use of CIR model for interest rate modeling in the Pakistani Financial markets

Abstract

This paper uses the equilibrium model of the term structure of interest rates based on the framework of Cox, Ingersoll and Ross (CIR, 1985). The contributions of this paper to the literature are empirical and the purpose is to derive estimates of parameters that fit the Continuous time CIR model using the available discrete market data of the yield curve. CIR one-factor model allows the conditional mean (drift) and conditional variance (diffusion) to be functions of the current short rate. The parameters have been estimated over different data sets, time periods and estimation techniques.

Introduction

The short-term interest rate is one of the key financial variables in any economy. It is a target instrument that central banks use to implement monetary policy and an important economic indicator for regulators and governments. It is also a key variable for business since it forms the basis of floating rate loans and most of the financial instruments that can be used to manage interest rate risk.

Moreover, longer-term interest rates reflect, at least in part, the expected values of future short rates. The short rate is also a relevant input in determining the required return on any asset. For all of these reasons, a good model of the short-term interest rate is of great practical importance.

There are two major classes of interest rate models. One class of term structure models called the short-rate models (Vasicek, Cox-Ingersoll-Ross and CKLS), the term structure is produced by the model. This can permit arbitrage if used by a dealer since traders can execute transactions at market prices, which differ from the dealer's prices. The parameters of these models are usually estimated by fitting the model to historical data. The entire term structure of interest rates can be inferred from the short rate process (Vasicek, 1977 and Cox, Ingersoll, and Ross, 1985).

The second class of models, called the no-arbitrage models, uses the term structure as an input. In other words in these models, one starts with the prices of zero coupon bonds of various maturities and proceeds to build a model that admits no arbitrage possibilities, while modeling how bond prices and interest rates evolve through time. No-arbitrage term structure models develop a time-series process for the short rate that is consistent with the current term structure of interest rates. That is, a short rate process is developed to "fit" the current term structure. The advantage of this class of models is that arbitrage against the current term structure of interest rates is impossible. The disadvantage is that the modeled short rate process might not reflect some of the key features of the empirically observed time series properties of interest rates. Examples of this type of model include Ho and Lee (1985), Hull and White (1990), Black and Karasinski (1991), Black, Derman, and Toy (1990), and Heath, Jarrow, and Morton (1992).

Most of the popular short rate models incorporate two key features: mean reversion and non-constant volatility. In particular, most models allow for the short rate to revert to a long-run mean so that if the current rate is above (below) the long-run mean it is expected to decrease (increase) towards the long-run mean in the future. The volatility of interest rates is often made to be dependent on the level of interest rates so that when rates are higher, they are more volatile.

In CIR model, the conditional mean (drift) and conditional variance (volatility) of the short rate are functions of the current level of the short rate.

The existing literature contains a number of papers that seek to estimate the parameters of the short rate models. Different authors use different data sets, time periods, sampling frequencies, and empirical methodologies. In this paper, the focus is on estimating the parameter for the Pakistani rupee. The yield on 3-m Treasury bill is used as the proxy for short rate.

The empirical methodology used for estimating parameters is the LSE (Least Square Estimate). Various researches on the subject have revealed that the parameter estimates are sensitive to the choice of empirical methodology.

This study is aimed at using the PKR interest rates and estimating the parameters for the CIR model.

Pakistan's Interest Rate Scenario

Pakistan's participation in the "war on terror" resulted in aid dollars and military hardware flow in profusion. The US waived $1 billion of debt and promised $3 billion over five years. Additionally the influx of remittances post 9/11 resulted in liquidity in the economy. This resulted in lower short term interest rates.

From 12.88% in June 2001 to 1.66% in June 2003, the yield on T-Bills were reduced in order to increase lending to private sector for long term economic growth. However, the overall decline in the interest rate structure of Pakistan was consistent with the decline in global interest rates.

The 6-months T-Bill yield declined significantly during the period from 2001 to 2004, reflecting the presence of massive liquidity in the banking system. The 6-month T-bill yield was as high as 12.88 percent in June 2001, declined sharply to 1.66 percent in June 2003 but move slightly upward to 2.08 percent in June 2004 – a decline of 1080 basis points in 3 years.

However, inflation picked up in 2004. A sharp increase in commodity prices and an unprecedented increase in international oil prices lead to an inflationary pressure across the globe. To contain the inflationary pressures, SBP changed its stance from an easy monetary policy to a 'measured' tightening of the monetary policy. The yield of the 6-months T-bills rose by 111 basis points to 3.19 percent. This resulted in an increase in the short term interest rates. The rise in interest rate was consistent with global rise in interest rate.

Since March 2005 until December 2005, the benchmark 6-moths T-bill interest rate had increased by 274 basis points (bps). However, since March 2004 the 6-months T-bill interest rate had increased by 650 bps. The discount rate was increased by 150 bps to 9.0 percent in April 2005 as part of the tight monetary policy. The tightening of the monetary policy has resulted in containing the inflation and steadying the short rates.

The 1180 days of 3-month T-bill rates data used as a proxy for short rates, exhibit the trend shown in Fig 1, Appendix A.

The Cox, Ingersoll, and Ross Model

Interest rate curves are a very useful way of visualizing the term structure of interest rates and are a necessary input to effective term structure analysis. The creation of these curves from market data involves contending with incomplete data, noisy data, differing market conventions, the misalignment of cash flows in time, and other issues.

A general equilibrium model is suitable for the Pakistani financial markets as the spot rate curve is an implied curve that is indicative and allows for arbitrage. Hence, the model selected is CIR model.

The model of Cox, Ingersoll, and Ross (1985) has linear mean reversion model and uses a diffusion process different than other short rate models. The CIR model states that the short rate follows a square root diffusion process, which has the following continuous-time representation:

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}\,dZt$$

where k represents the speed of adjustment (or mean reversion),
θ represents the long run mean of the short-term interest rate, and
σ represents the volatility.

Under this model, both the drift and the volatility change with the level of the short rate. It has the same mean reversion feature as the Vasicek model; however the stochastic term has a standard deviation proportional to the square root of the current short rate. This implies that as the short rate increases, its standard deviation increases. Moreover, the CIR model has the conceptual advantage that the short rate will be strictly non-negative. As the short rate falls and approaches zero, the diffusion term (which contains the square root of the short rate) also approaches zero. In this case, the mean-reverting drift term dominates the diffusion term and pulls the short rate back towards its long-run mean. This prevents the short rate from falling below zero.

The CIR model has been employed for the purpose of modeling the PKR term structure of interest rates, using the 3 month T-bill rates as the short term rate. Additively, the spread between the 3 month T-bill rates and the 1 year T-bill rates, the spread between the 1 year T-bill rates and 5 year T-bonds (PIB) and the spread between the 5 year T-bonds and the 10 year T-bonds have been modeled using CIR, in order to capture the volatility of spreads in the longer maturities. The term structure is reconstructed using these model-generated rate and spreads.

Data Selection

The parameters are being estimated for the CIR short rate model for Pakistani rupee (PKR). The three-month T-bill rate is used as a proxy for the short rate, which is a liquid instrument in Pakistani Financial market. Daily observations from 7 June 2003 to 31 March 2006. For weekends and holidays, the last available rate has been replicated.

To construct a yield curve, the spread between 3-month and 1-year T-bill rate, the spread between 1-year T-bill rate and 5-year T-bonds (Pakistan Investment bonds) and the spread between 5-year T-bonds (Pakistan Investment bonds) and 10-year T-bonds (Pakistan Investment bonds) have been considered for parameter estimation.

The short rate (3-monthe rate) ranges from 0.83% p.a on 11^{th} August 2003 to 8.33% on 31 March 2006 The short rates have a mean of 4.1171% p.a and are negatively skewed and exhibit excess kurtosis. The 1-yr spreads have a mean of 0.5071% p.a, 5-yr spreads have a mean of 1.7965% p.a and 10-yr spread have a mean of 1.0185% p.a.

When using discrete data for estimating the parameters of a continuous model, one needs a discrete representation of the process. Griselda Deelstra and Gary Parker (1995), use the principle of covariance equivalence to find a discrete representation of the Cox-Ingersoll-Ross model. This principle requires that the first two moments of both the discrete and the continuous models be equal.

Deelstra and Parker (1995) obtain the expected value and autocovariance function of centered CIR continuous time model for short rates. The following transformation was applied to the CIR model to obtain the centered CIR model:

$$r_t = r_t^* - \gamma .$$

where

r_t^* = spot rate
γ = the long-term mean

The centered CIR model used was

$$dr_t = -\kappa r_t dt + \sigma\sqrt{r_t + \gamma}\, dB_t$$

The expected value followed as:

$$E[r_t] = e^{-\kappa t} r_0$$

The covariance function was

$$\operatorname{cov}(r_t, r_s) = \sigma^2 \frac{e^{-\kappa t} - e^{-\kappa(s+t)}}{\kappa} r_0 + \sigma^2 \frac{e^{-\kappa(t-s)} - e^{-\kappa(t+s)}}{2\kappa} \gamma, \quad s \le t$$

And the stationary variance was found to be

$$\lim_{t \to \infty} \operatorname{Var}[r_t] = \frac{\gamma \sigma^2}{2\kappa}$$

They used the principle of covariance equivalence as described by Pandit and Wu (1983) and used for Gaussian first and second order stochastic differential equations by Parker (1995) . They considered two discrete models as possible sampled representations of the Cox-Ingersoll-Ross model. They also established parametric relationship between the discrete and continuous models.

One was the simple discretisation model, found in the literature and the other was the covariance equivalent discretisation proposed by Deelstra and Parker.

The **Simple Discretisation** function used was

$$r_t = \phi r_{t-1} + \sigma_a \sqrt{r_{t-1} + \gamma}\, a_t, \qquad t = 1, 2, 3, \ldots$$

The expected value, covariance and auto covariance functions were

$$\mathbf{E}[r_t] = \phi^t r_0$$

$$\operatorname{cov}(r_t, r_s) = \phi^{t-1} \sigma_a^2 r_0 \frac{1 - \phi^s}{1 - \phi} + \phi^{t-s} \sigma_a^2 \gamma \frac{1 - \phi^{2s}}{1 - \phi^2}, \quad s \le t.$$

$$\lim_{t \to \infty} \operatorname{Var}[r_t] = \frac{\sigma_a^2 \gamma}{1 - \phi^2}$$

Although explicit parametric relationships between the continuous and discrete process were obtained

$$\phi = e^{-\kappa}$$

$$\sigma_a^2 = \sigma^2 \frac{1 - \phi^2}{2\kappa} = \sigma^2 \frac{1 - e^{-2\kappa}}{2\kappa}$$

A substitution of these relations in the co-variances of continuous and discrete models revealed that the co-variances are not equal for all s and t.

The **Covariance Discretisation** function proposed by Deelstra and Parker (1995) was

$$r_t = \phi r_{t-1} + \sigma_a \sqrt{\frac{2\phi}{1+\phi} r_{t-1} + \gamma}\; a_t \qquad t = 1,2,3,\ldots$$

The expected value, covariance and auto covariance functions were

$$\mathbf{E}[r_t] = \phi^t r_0$$

$$\operatorname{cov}(r_t, r_s) = 2\phi^t \sigma_a^2 r_0 \frac{1-\phi^s}{1-\phi^2} + \phi^{t-s}\sigma_a^2 \gamma \frac{1-\phi^{2s}}{1-\phi^2}, \quad s \le t$$

$$\lim_{t\to\infty} \operatorname{Var}[r_t] = \frac{\sigma_a^2 \gamma}{1-\phi^2}$$

Although explicit parametric relationships between the continuous and discrete process were obtained

$$\phi = e^{-\kappa}$$

$$\sigma_a^2 = \sigma^2 \frac{1-\phi^2}{2\kappa} = \sigma^2 \frac{1-e^{-2\kappa}}{2\kappa}.$$

When the parametric relationships were substituted, the covariance of continuous and discrete process was found to be equal.

The two discrete models produce different results when used as approximations of the CIR model. In practice, the difference is even compounded by the fact that a given series of data will produce different parameter estimates depending on the discrete model used.

Deelstra and Parker (1995) established the parametric relations between the continuous and the discrete models and than used least squares estimates of the parameters of the discrete models.

Estimation of parameters using Least Square Method

The parameters of a discrete representation of the CIR model are first estimated. The least squares method was used to estimate the parameters of the discrete model.

The least squares estimate of Φ is the value that minimizes the residual sum of squares,

$$RSS = \sum_{t=1}^{N} \sigma_a^2 a_t^2$$

where N is the number of observations and the residuals, $\sigma_a^2 a_t^2$, are calculated from the data according to the specific model used.

For the simple discretisation,

$$\sigma_a^2 a_t^2 = \frac{\left(r_t - \phi\, r_{t-1}\right)^2}{r_t + \gamma}.$$

For the covariance equivalent discretisation,

$$\sigma_a^2 a_t^2 = \frac{\left(r_t - \phi\, r_{t-1}\right)^2}{\frac{2\phi}{1+\phi} r_t + \gamma}$$

Finally, the least squares estimate of σ_a^2 is given by RSS/(N- 1).

These discrete parameters are than transformed into the parameters for the continuous CIR model using the parametric relationships discussed earlier.

Results

The two methods used for deriving the parameters were applied to the data consisting of annualized yields on 3-month T-bills and on spreads between four tenors. The historical data used was for 1180 days from 7 January 2003 to 31 March 2006.

The results for the estimation of parameters using the two methods are given in Appendix B. The parameters were estimated for different number lengths of time from 1180 days, 600 days, 400 days, 300 days, 200 days to 100 days.

The estimates of κ were found to be different when Covariance Discretisation model was used instead of the Simple Discretisation model. The estimated parameters are expressed in daily units as daily data was used. The volatilities arrived at through the parametric relationship in case of both the models show very little difference. However, it is observed that as that data set being analyzed is narrowed, the drift and volatility estimates differ relatively more between the two model used.

Conclusion

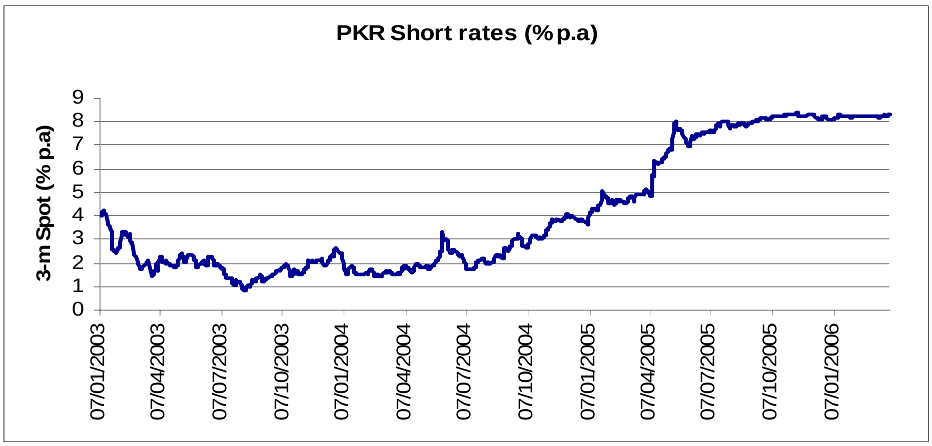
In this study, the simple discretisation method was compared with the Covariance Discretisation method of the CIR model. Using the market data and the parametric relationships established by Deelstra and Parker (1995), the parameters for the CIR model were estimated for the PKR interest rates.

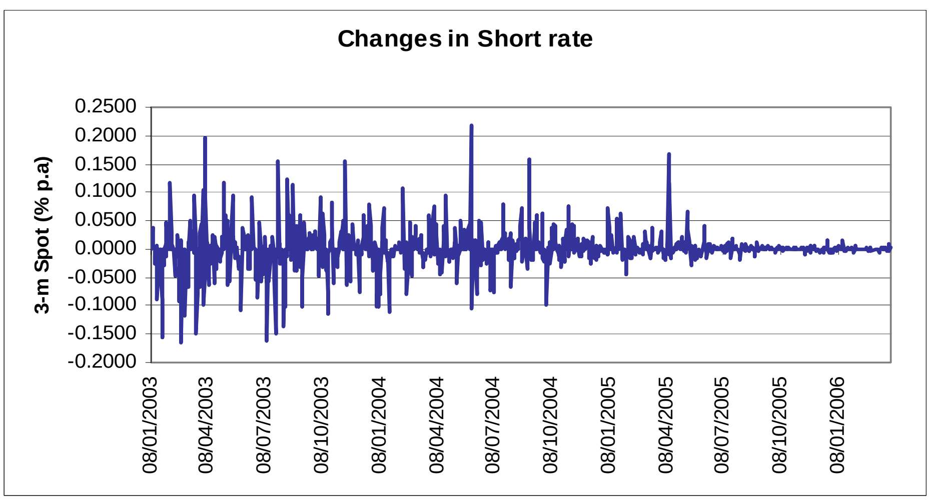
Since, the Covariance Discretisation model has the same first two moments as the CIR model at all sampling times, it is suggested by Deelstra and Parker (1995) to use this model as a discrete representation of the Continuous CIR model.

The estimates facilitate the use of CIR model for the PKR interest rates and help in the pricing and valuations of the derivative products yet in a state of infancy in Pakistan. As the market for derivative products evolves, the use of alternative term structure models would become an obvious choice.

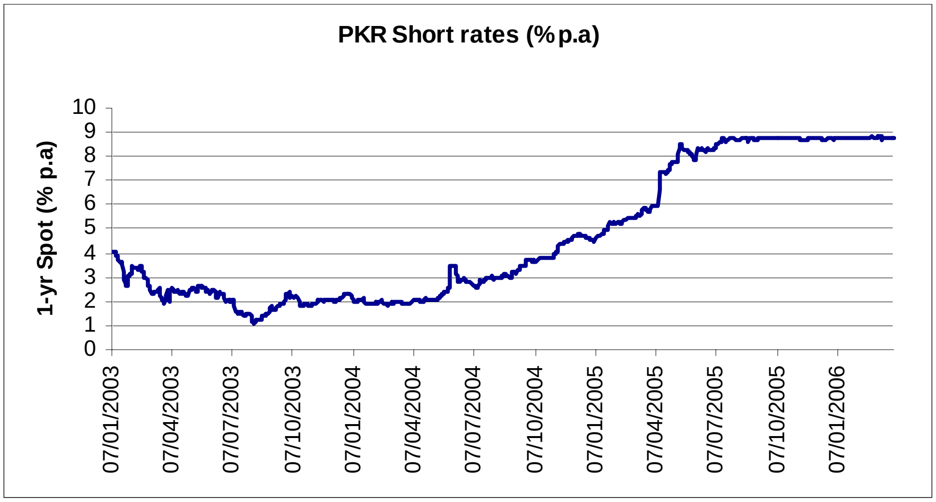
Appendix A

3 month rates





1-Year Rates


PKR Short rates (% p.a)
1-yr Spot (% p.a)
10
9
8
7
6
5
4
3
2
1
0
07/01/2003
07/04/2003
07/07/2003
07/10/2003
07/01/2004
07/04/2004
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07/10/2004
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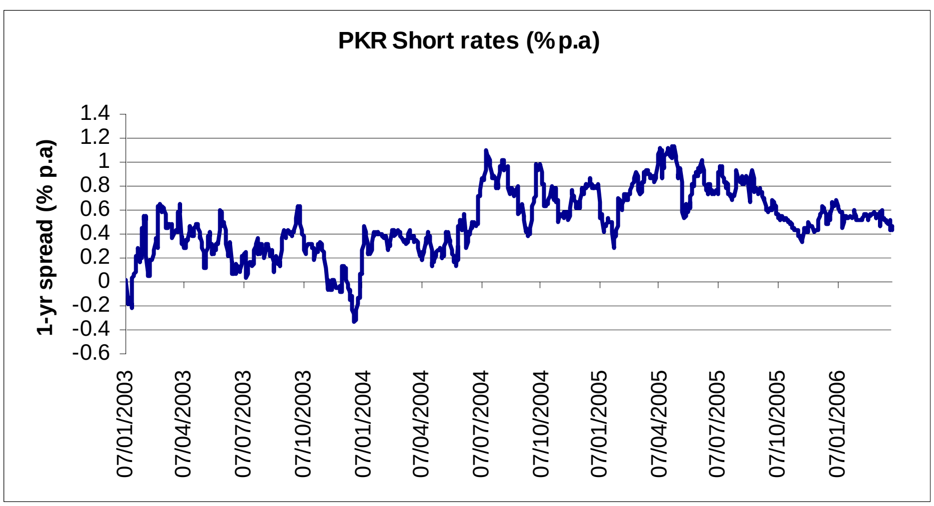
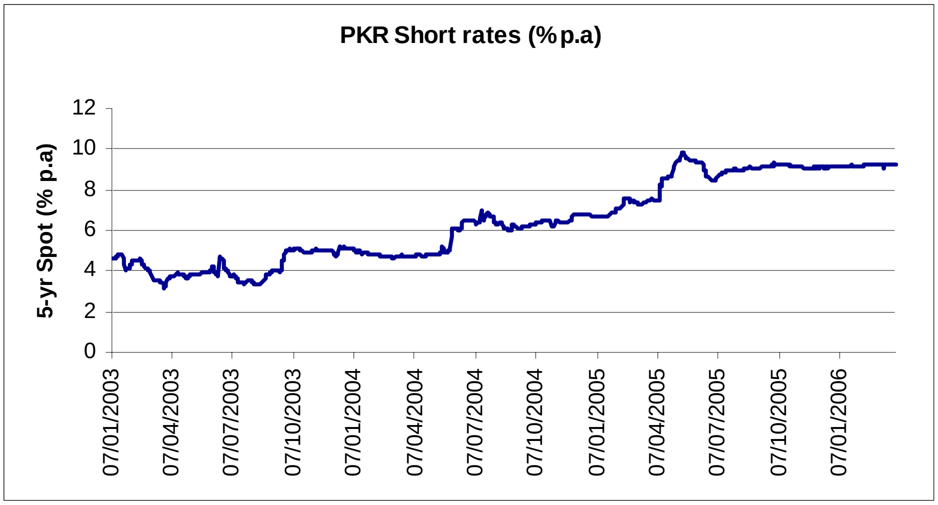
07/07/2005
07/10/2005
07/01/2006


PKR Short rates (% p.a)
1-yr spread (% p.a)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
07/01/2003
07/04/2003
07/07/2003
07/10/2003
07/01/2004
07/04/2004
07/07/2004
07/10/2004
07/01/2005
07/04/2005
07/07/2005
07/10/2005
07/01/2006

5-Year Rates


PKR Short rates (% p.a)
5-yr Spot (% p.a)
12
10
8
6
4
2
0
07/01/2003
07/04/2003
07/07/2003
07/10/2003
07/01/2004
07/04/2004
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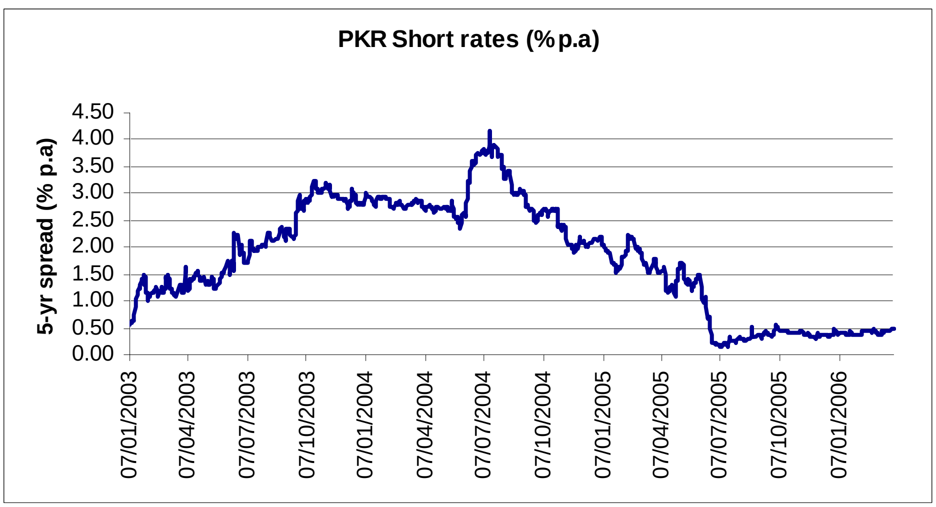
07/07/2005
07/10/2005
07/01/2006


PKR Short rates (% p.a)
5-yr spread (% p.a)
4.50
4.00
3.50
3.00
2.50
2.00
1.50
1.00
0.50
0.00
07/01/2003
07/04/2003
07/07/2003
07/10/2003
07/01/2004
07/04/2004
07/07/2004
07/10/2004
07/01/2005
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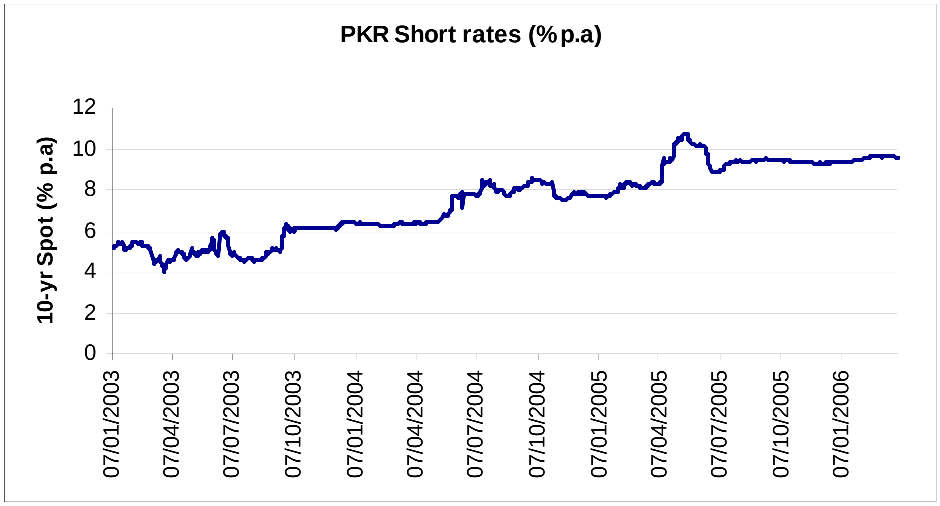
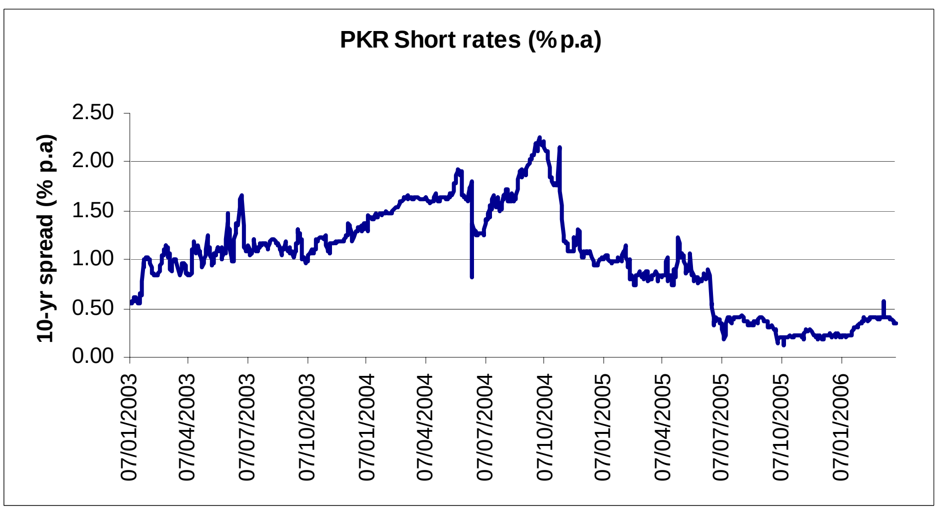
07/07/2005
07/10/2005
07/01/2006

10-Year Spread





Appendix B

(1180 days) 39 months of PKRV Data --- 7th Jan 2003 to 31st Mar 2006				
Simple Discretisation				
	PKRV 90 Days	**Spread 1 yr**	**Spread 5 yr**	**Spread 10 yr**
γ (long term mean)	4.117127%	0.507122%	1.796454%	1.018464%
Φ (drift-discrete)	0.998688217	0.984316528	0.999637633	0.997460794
σ_a (vol-discrete)	0.76747%	0.009185163	0.005757582	0.006090559
κ (drift)	0.13126%	1.58078%	0.03624%	0.25424%
σ (Volatility)	0.76798%	0.92579%	0.57586%	0.60983%
Covariance Discretisation				
	PKRV 90 Days	**Spread 1 yr**	**Spread 5 yr**	**Spread 10 yr**
γ (long term mean)	4.117127%	0.507122%	1.796454%	1.018464%
κ (drift)	0.15254%	1.58077%	0.07908%	0.33098%
σ (Volatility)	0.76798%	0.925785%	0.57591%	0.60987%

600 days of PKRV Data --- 9th Aug 2004 to 31st Mar 2006				
Simple Discretisation				
	PKRV 90 Days	**Spread 1 yr**	**Spread 5 yr**	**Spread 10 yr**
γ (long term mean)	6.21%	0.69%	1.24%	0.78%
κ (drift)	0.50239%	2.75014%	0.10143%	0.22813%
σ (Volatility)	0.48812%	0.53371%	0.55966%	0.53846%
Covariance Discretisation				
	PKRV 90 Days	**Spread 1 yr**	**Spread 5 yr**	**Spread 10 yr**
γ (long term mean)	6.205550%	0.686767%	1.239683%	0.781378%
κ (drift)	0.51327%	2.96201%	0.16212%	0.30631%
σ (Volatility)	0.48802%	0.53376%	0.55965%	0.53847%

400 days of PKRV Data --- 9th Aug 2004 to 31st Mar 2006				
Simple Discretisation				
	PKRV 90 Days	**Spread 1 yr**	**Spread 5 yr**	**Spread 10 yr**
γ (long term mean)	6.21%	0.69%	1.24%	0.78%
κ (drift)	0.50239%	2.75014%	0.10143%	0.22813%
σ (Volatility)	0.48812%	0.53371%	0.55966%	0.53846%
Covariance Discretisation				
	PKRV 90 Days	**Spread 1 yr**	**Spread 5 yr**	**Spread 10 yr**
γ (long term mean)	6.205550%	0.686767%	1.239683%	0.781378%
κ (drift)	0.51327%	2.96201%	0.16212%	0.30631%
σ (Volatility)	0.48802%	0.53376%	0.55965%	0.53847%

300 days of PKRV Data --- 9th Aug 2004 to 31st Mar 2006				
Simple Discretisation				
	PKRV 90 Days	**Spread 1 yr**	**Spread 5 yr**	**Spread 10 yr**
γ (long term mean)	6.21%	0.69%	1.24%	0.78%
κ (drift)	0.50239%	2.75014%	0.10143%	0.22813%
σ (Volatility)	0.48812%	0.53371%	0.55966%	0.53846%
Covariance Discretisation				
	PKRV 90 Days	**Spread 1 yr**	**Spread 5 yr**	**Spread 10 yr**
γ (long term mean)	6.205550%	0.686767%	1.239683%	0.781378%
κ (drift)	0.51327%	2.96201%	0.16212%	0.30631%
σ (Volatility)	0.48802%	0.53376%	0.55965%	0.53847%

200 days of PKRV Data --- 9th Aug 2004 to 31st Mar 2006				
Simple Discretisation				
	PKRV 90 Days	**Spread 1 yr**	**Spread 5 yr**	**Spread 10 yr**
γ (long term mean)	6.21%	0.69%	1.24%	0.78%
κ (drift)	0.50239%	2.75014%	0.10143%	0.22813%
σ (Volatility)	0.48812%	0.53371%	0.55966%	0.53846%
Covariance Discretisation				
	PKRV 90 Days	**Spread 1 yr**	**Spread 5 yr**	**Spread 10 yr**
γ (long term mean)	6.205550%	0.686767%	1.239683%	0.781378%
κ (drift)	0.51327%	2.96201%	0.16212%	0.30631%
σ (Volatility)	0.48802%	0.53376%	0.55965%	0.53847%

100 days of PKRV Data --- 9th Aug 2004 to 31st Mar 2006				
Simple Discretisation				
	PKRV 90 Days	**Spread 1 yr**	**Spread 5 yr**	**Spread 10 yr**
γ (long term mean)	6.21%	0.69%	1.24%	0.78%
κ (drift)	0.50239%	2.75014%	0.10143%	0.22813%
σ (Volatility)	0.48812%	0.53371%	0.55966%	0.53846%
Covariance Discretisation				
	PKRV 90 Days	**Spread 1 yr**	**Spread 5 yr**	**Spread 10 yr**
γ (long term mean)	6.205550%	0.686767%	1.239683%	0.781378%
κ (drift)	0.51327%	2.96201%	0.16212%	0.30631%
σ (Volatility)	0.48802%	0.53376%	0.55965%	0.53847%

References:

Hull, John (1989).
Options, Futures, and other derivative securities: 274-276

Deelstra, Griselda and Parker, **Gary** (1995).
A Covariance Equivalent Discretisation of the. CIR Model. Proceedings of the V AFIR Colloquium

K.C. Chan, G. Andrew Karyoli, Francis A. Longstaff and Anthony B. Sanders
(1992)
The Volatility of Short-Term Interest Rates: An Empirical Comparision of Alternative Models of the Term Structure of Interest rates